The stable problem in the Rindler space-time *

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Abstract

We carefully study the stable problem of the Rindler space time by the scalar wave perturbation. Using the two different coordinate systems, the scalar wave equation is investigated. The results are different in these two cases. They are analyzed and compared in detail. The conclusions are: (a) the Rindler space time as a whole is not stable; (b) the Rindler space time could exist stably only as a part of the Minkowski space time, and the Minkowski space time could be a real entity independently; (c) there are some defects for the scalar wave equation written by the Rindler coordinates, and it is unsuitable for investigation of the stable properties of the Rindler space time. All these results might shed some lights on the stable properties of the Schwarzschild black hole. It is natural and not unreasonable for one to infer that: (a) perhaps the Regge-Wheeler equation is not sufficient to decide the stable properties; (b) the Schwarzschild black hole as a whole might be really unstable; (c) the Kruskal space time is stable and can exist as a real physical entity ; whereas the Schwarzschild black hole could occur only as part of the Kruskal space time.

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I. Motivation

The stability of the Schwarzschild black hole is a vitally important problem in general relativity. In 1957, Regge and Wheeler first treated the problem by the Schwarzschild coordinates [1]. They obtained the perturbation equation, that is, the well-known Regge-wheeler equation. The horizon and the spatial infinity consist of the boundaries for the perturbation problem. It is easy to treat the boundary of the spatial infinity; Whereas the boundary conditions at the horizon are complicated. The background metric has an apparent singularity at the horizon $r = 2m$ in the Schwarzschild coordinates; this in turn may result in spurious divergence in the perturbation and cause confusion. A proposition was given by Vishveshwara in 1970. Because the metric is regular at $r = 2m$ in the Kruskal coordinates system, he transformed the perturbation fields into it. By using the Schwarzschild time $t = 0$ to define the initial time, it is obtained that the black hole is stable [2].

In practice, there is still another problem concerned with the Schwarzschild coordinates. Due to its metric component $g_{00} = 0$ at the horizon $r = 2m$, the Schwarzschild time coordinate t loses its meaning at the horizon. This is the drawback of the Schwarzschild time coordinate t. For example, a particle near the horizon $r = 2m$ falls into the black hole by finite proper time; but it is described by the Schwarzschild time coordinate as an infinity process as $t \to +\infty,\ r \to 2m$. This example shows that the Schwarzschild time coordinate t is a tortoise coordinate. It is well-known that $r_* = r + 2m \ln(\frac{2m}{r} - 1)$ is a tortoise coordinate.

Actually, this tortoise property of t appears more easily to escape one's notice, and the researchers had hardly considered its influence on the stable problem[1],[2]: $t = 0$ is taken

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it for granted as the initial time at the horizon. One should doubt its validity. We have first taken notice of it and worked over it. For simplification, one might select some "good" time coordinate to define the initial time. Using the new "good" coordinate time, the stable problem should be reinvestigated again. The Kruskal time T and the Painlevé time are all "good" coordinate times. We have employed them to study the problem and obtained unusual results. By using the Kruskal time coordinate $T = C = const$ to define the initial time, we found that its stable properties controversially depend on the sign of the initial time $T = C = const$: the Schwarzschild black hole is stable when $T = C \geq 0$, whereas the Schwarzschild black hole is unstable when $T = C \leq 0$[3]-[6]. The same is also true for the Painlevé time: when the initial time slice intersects the future horizon, it is stable; whereas it is unstable as the initial time slice intersects the past horizon. These unexpected results are in contrast with the conclusion taken it for granted that the Schwarzschild black hole is stable.

Which one is correct? Whether is the Schwarzschild black hole stable or not? Where and why do the contradictories come from? Is it just an effect caused by the tortoise property of the time coordinate t? Actually, we have already replaced it by "good" time, this by turn excludes this possibility. Perhaps this complicated situation really shows that the Regge-Wheeler equation, which was obtained in the Schwarzschild coordinates, is not sufficient to decide the stable properties. As shown before, the Schwarzschild coordinate t is a tortoise coordinate as the well-known tortoise coordinate r_*; they do not cover the event horizon. So, they are possibly not qualified for the stable study.

The Kruskal space time is the extension of the Schwarzschild black hole. The Kruskal coordinates are qualified for study of the perturbation equation and the stable problem. But the background metric is varying with time in the Kruskal coordinates [2], subsequently, it is almost impossible to restudy the problem in the whole Kruskal coordinates. So there is still not a convincing answer to the stable problem of the Schwarzschild black hole.

On the ground of the difficulties, we want to study the stable problem of the Schwarzschild black hole indirectly and are led to study that of the Rindler space time.

The Rindler space time is one part of the Minkowski space time whose constant spatial coordinates describe accelerated observers with constant accelerations. The interest in Rindler space time lies in almost the same geometrical structure with that of the Schwarzschild black-hole. Actually, the Hawking radiation in the Schwarzschild black-hole is closely connected with the Unruh effect in the Rindler space time[7],[8].

The Rindler coordinates are the same as those of the Schwarzschild time coordinates; its time is a tortoise coordinate too. What effect would it have on the stable study of the Rindler space time? In order to find it, we will study the Klein-Gordon equation in the whole Rindler coordinates in section II and discuss the stable properties of the Rindler space time. The same problem is treated in section III by the Minkowski coordinates; some arguments are given and conclusion is obtained in section IV.

II.The scalar field equation in the Rindler space time

The metric of the Rindler space time is

$$ds^2 = -z^2 dt^2 + dz^2 + dx^2 + dy^2 \tag{1}$$

with $0 < z < +\infty,\ t,\ x,\ y \in (-\infty,\ +\infty)$. The Minkowski space time's metric is

$$ds^2 = -dT^2 + dZ^2 + dx^2 + dy^2. \tag{2}$$

The transformation equations between them are

$$z = \sqrt{Z^2 - T^2} \tag{3}$$

and

$$e^{2t} = \frac{Z+T}{Z-T}, \tag{4}$$

or

$$Z = z\cosh t \tag{5}$$

and

$$T = z\sinh t. \tag{6}$$

Of course, the Rindler space time's completion is the Minkowski space time, and corresponds to only the part $Z > 0$, $Z^2 - T^2 > 0$ of the Minkowski space time. $Z^2 - T^2 = 0$ is the horizon of the Rindler space time. The horizon could be denoted alternatively by $z = 0$ in the Rindler coordinates. The metric (1) is obviously singular at the horizon.

In the Rindler coordinates, the Klein-Gordon equation of the scalar field with mass μ is

$$\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}\left[\sqrt{-g}g^{\mu\nu}\frac{\partial \Psi}{\partial x^\nu}\right] - \mu^2\Psi = 0, \tag{7}$$

which turns out as

$$-\frac{1}{z}\frac{\partial^2\Psi}{\partial t^2} + \frac{1}{z}\frac{\partial}{\partial z}\left[z\frac{\partial\Psi}{\partial z}\right] + \frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} - \mu^2\Psi = 0. \tag{8}$$

The coordinates $t,\ x,\ y$ range from $-\infty$ to $+\infty$, therefore, the normal mode decomposition of the scalar field Ψ must be

$$\Psi = \psi(z)e^{-i\omega t + ik_1 x + ik_2 y} \tag{9}$$

with ψ satisfying the following equation

$$\frac{d}{dz}\left[z\frac{d\psi}{dz}\right] - \frac{\omega^2}{z}\psi + z\left[\mu^2 + k_1^2 + k_2^2\right]\psi = 0. \tag{10}$$

The equation (10) is Bessel's equation with its order being $\pm i\omega$[8]. Its solutions are the modified Bessel function $I_{\pm i\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\ z)$.[1] Suppose the frequency $\omega = \alpha + i\beta$ with $\alpha,\ \beta$ real and $\beta > 0$, then $I_{-i\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\ z)$ goes to infinity exponentially as $z \to \infty$ and vanishes at $z = 0$. On the contrary, $I_{+i\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\ z)$ goes to infinity at $z = 0$ and falls off to zero exponentially at $z \to \infty$[8].

The boundaries of the equation (10) consist of $z = 0$, the horizon $H^\pm$ of the Rindler space time, and $z \to \infty$ of the infinity. We must demand the scalar field function Ψ initially well-behaved at the boundaries. For the infinity, we could require the field Ψ falling off to zero initially. So we just select $I_{i\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\ z)$ to satisfying the infinity boundary condition initially with the frequency whose imaginary is positive, that is,

$$\Psi = e^{-i\omega t + ik_1 x + ik_2 y} I_{i\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\ z). \tag{11}$$

Generally, researchers would use the Rindler coordinate time $t = 0$ as the initial time and study the problem. By Eq.(11), the scalar field is regular at the infinity ($z \to \infty$)at $t = 0$. But $I_{i\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\ z)$ blows up at the horizon $z = 0$ at $t = 0$, that is, Ψ is initially blows up at the horizon for the positive imaginary of the frequency ($\beta > 0$). This shows that $\beta > 0$ is unacceptable for the scalar field because it behaves badly at the initial time $t = 0$. Therefore the scalar field is stable and the Rindler space time is also stable to some extent (confining to the perturbation of the scalar field).

But the metric is singular at the horizon $z = 0$, and the time coordinate t even loses its meaning there due to the fact $g_{00}|_{z=0} = 0$. In reality, the past and future horizons correspond

[1] when $i\omega$ is an positive integer, $I_{-i\omega}$ is replaced by $K_{-i\omega}$.

to $t \to -\infty$,$t \to +\infty$ respectively. Their intersection is $T = Z = 0$ in Minkowski coordinates, but it corresponds to $z = 0$ and t from $-\infty$ to $+\infty$ in the Rindler coordinates. This example clearly shows the Rindler time coordinate t is meaningless and $t = 0$ should not be used as initial time containing the horizon. Therefore, we can not be sure that the above stable conclusion is really right; we should also employ the Minkowski time coordinate to treat the problem again.

Now, we make use of the Minkowski time T to define the initial time. By the transformation equations (3),(4), we could rewrite the scalar field Ψ as the function of the independent variables $T,\ Z,\ x,\ y$ as:

$$\Psi = Ae^{\frac{-i\omega t}{2}\ln\frac{Z+T}{Z-T}}e^{ik_1x+ik_2y}I_{i\omega}(\sqrt{\mu^2+k_1^2+k_2^2}\sqrt{Z^2-T^2}). \tag{12}$$

Excluding the horizon, the scalar field Ψ is bounded at the initial time $T = C = const$ for the positive frequency $\Im\omega = \beta > 0$. At the horizon $Z^2 - T^2 = 0$, the asymptotic form of the scalar field Ψ in Eq.(12) is

$$\Psi \propto A\left(\frac{Z+T}{Z-T}\right)^{\frac{-i\omega}{2}}\left(Z^2-T^2\right)^{\frac{i\omega}{2}} = A\,(Z-T)^{i\omega}\,. \tag{13}$$

Whether or not the equation (13) is initially divergent at the horizon really depends on the sign of the initial time $T = C$. When we select the initial time $T = C \geq 0$, The initial time slice $T = C \geq 0$ intersects the future horizon that corresponds to $Z = T = C$. So, the scalar field Ψ goes to infinity initially at $Z = T = C$ (the horizon). This excludes the possibility of the positive imaginary frequency, and subsequently ensures its stability and the Rindler space time respectively to some extent.

Alternatively, we could also select the initial time $T = C < 0$. The initial time slice $T = C$ intersects the past horizon that is denoted by $Z = -T = -C$, so the scalar field Ψ is also bounded initially at the horizon for the positive imaginary frequency $\beta > 0$. In this very case, the scalar field Ψ is not stable and the Rindler space time consequently is also unstable with respect to the scalar field perturbation.

Here comes some discussion.

It has been shown that the stable properties of the Rindler space time vitally depend on using what time coordinate to define the initial time. If we use the Rindler time, the scalar field is stable. On the ground of that, the Rindler time coordinate loses its time meaning in the horizon, one should choose "good" time to define the initial time. Of course, the Minkowski time is one of the "good" time. When we choose the Minkowski time to define the initial time, the scalar field' stable properties become very intricate and depend on where the initial time slice intersects the horizon. The Rindler space time is stable when the initial time slice intersects the future horizon; while it is unstable as the initial time slice intersects the past horizon. The controversial conclusions is almost the same as those for the Schwarzschild black hole [3]-[6]. This problem is unsolved in the Schwarzschild black hole. It is suggested that one should study the stable problem of the Schwarzschild black hole in the whole Kruskal coordinate system to get a definite answer to the controversy[2]. But it becomes almost impossible because the background metric in the Kruskal coordinate system is not even stationary. So, one has not obtained an absolutely and convincingly unambiguous answer to the stable problem of the Schwarzschild black hole.

The results are similar in the two cases. This is a truly good news. We could really get something interesting for the Schwarzschild black-hole from the present results. The Minkowski space time is the extension of the Rindler space time, and is luckily static in nature. This static property of the Minkowski spcae time presents a striking contrast to that of the whole Kruskal space time, i.e., the extension of the Schwarzschild black hole. We could easily investigate the perturbation equation of the Rindler space time in whole Minkowski coordinates. It is not difficult to obtain the result. It might give some clues to the stable problem of the Schwarzschild black hole.

III. The scalar field equation in the Minkowski space time

The scalar field equation in the Minkowski space time is

$$-\frac{\partial^2\Psi}{\partial T^2}+\frac{\partial^2\Psi}{\partial Z^2}+\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}-m^2\Psi=0. \tag{14}$$

The Rindler space time corresponds the region where the coordinates $T,\ x,\ y$ range from $-\infty$ to $+\infty$ and $Z>0,\ Z^2-T^2>0$, therefore, the normal mode decomposition of the scalar field Ψ must be

$$\Psi=Ae^{-i\omega T+ik_3Z+ik_1x+ik_2y} \tag{15}$$

where k_3 must satisfy the following equation

$$k_3=\pm\sqrt{\omega^2-k_1^2-k_2^2-m^2}. \tag{16}$$

In order to make the scalar field Ψ initially finite over $-\infty<x,\ y<+\infty$, the numbers $k_1,\ k_2$ of course are real. If we similarly choose the frequency $\omega=i\beta$ with $\beta>0$, the above equation becomes

$$k_3=\pm i\sqrt{\beta^2+k_1^2+k_2^2+m^2}. \tag{17}$$

We could select $k_3=i\gamma$ with $\gamma=\sqrt{\beta^2+k_1^2+k_2^2+m^2}>0$, then the scalar field Ψ is

$$\Psi=Ae^{\beta T}e^{-\gamma Z+ik_1x+ik_2y}. \tag{18}$$

The boundaries of the Rindler space time consist of the infinity $Z\to+\infty$ and the horizon $Z^2-T^2=0$. Because the Minkowski metric is regular everywhere including the horizon, we could directly use the Minkowski time $T=0$ to define the initial time. By Eq.(18), we could see the scalar field Ψ is well-behaved everywhere corresponding to the Rindler space time at the initial time $T=0$. But it blows up to infinity as the time T goes to infinity for $\beta>0$.

Directly, it is easy to see that the Rindler space time as a whole is unstable with respect to the scalar field perturbation.

IV. Arguments and conclusion

We just obtain the conclusive instability of the whole Rindler space time by the Minkowski coordinates. We now check the study in the Rindler coordinates in section II. The result definitely supports the view that the Rindler time coordinate t is not suitable for one to study the stable problem containing the horizon where t loses its meaning. This might suggest that the Schwarzschild time $t=0$ should not be qualified as initial time for study of the Regge-Wheeler equation in the Schwarzschild black hole.

The result also supports that only the instability in section II by using the Minkowski time $T=C<0$ is correct. This can be explained easily. The Minkowski space time consists of four parts: the right region R, i.e.,the Rindler space time $Z>0,\ Z^2-T^2>0$; the left region L, i.e.,$Z<0,\ Z^2-T^2>0$; the past region P,i.e., $T<0,\ Z^2-T^2<0$; the future region F,i.e., $T>0,\ Z^2-T^2<0$. At the initial time slice $T=C<0$, the Rindler space time could be influenced by the part P where every particle must leave and come into either the Rindler space time or the other parts L and F. In this way, the Rindler space time is unstable. Similarly, when the initial time slice intersects the past horizon of the Schwarzschild black hole, it it easy to see that its instability comes from the influence of the white hole. Though the Kruskal metric of the Schwarzschild black hole is not stationary, and one has not obtained the same conclusive result as that for the Rindler space time in the Minkowski coordinates; it is nevertheless not unreasonable to infer that the Schwarzschild black hole as a whole might really be unstable in comparison with the case in Rindler space time. Of course, one must go further to get the conclusion definitely.

Why is the stable conclusion wrong when using the Minkowski time $T = C > 0$? Furthermore, it is well-known that the Minkowski is stable, how does one harmonize our instability conclusion of the Rindler space time with this fact? Now we once again investigate the process for our conclusion. By the Eq.(18), only when the scalar field Ψ is completely confined to the Rindler space time, could it has positive imaginary frequency. This by turn shows that the Rindler space time as a whole is unstable. On the contrary, when the whole Minkowski space time is served as its background (Z ranges from $-\infty$ to $+\infty$), the Eq.(18) clearly excludes the possibility of the positive imaginary frequency of the scalar field Ψ. Therefore, the Minkowski space time is definitely stable. Actually, this is just the well-know fact of the Minkowski space time; no new conclusion comes. Really, there lies some inconsistency or contradiction in these conclusions. How to explain them? We give some naive arguments below.

When the Minkowski time $T = C > 0$ is the initial time, only the part F borders upon the Rindler space time initially. Classically, the part F could not influence the Rindler space time, but this might be false from the quantum theory. It is well-known from the de-Broglie's theory of matter wave that the phase velocity of the relativistic particles' might be possibly greater than that of the light. The scalar wave equation could also be regarded as a quantum wave equation for a freely moving particle with relativistic velocity. From the wave eqs.(10)-(13) and by the WKB method, it is not difficult to see that the phase velocity of the wave is not greater than that the light velocity at the horizon. So, the part F could not have any effect on the Rindler space time in quantum way (or by wave theory). This insures the stable conclusion. But now we know the conclusion is wrong, there must exist something wrong in the wave equation (10) or (8). To see this clearly, let's inspect the wave equations (14) and (15) in the Minkowski coordinates. The Eq.(15) presents a striking contrast to Eq.(10), its phase velocity can easily greater than the light velocity at the horizon. Therefore the part F has definitely influence on the Rindler space time, so does the part L. The Rindler space time as a whole is impossible to be stable, only the whole Minkowski space time is stable. From the above argument, we see that the wave equation (8) loses some useful information at the horizon and might result in error in the stable study for the Rindler space time.

Therefore, we reinforce our conclusion again: the whole Minkowski space time is a real entity, while the Rindler space time can not exists independently. Actually, the Rindler space time cannot exist if the left region L does not exist.

When compared with the stability study of the Schwarzschild black hole, one could reasonably infer the following: (1) the perturbation equation should be in the Kruskal space time, though it may be difficult; (2)the Regge-Wheeler equation is possibly not sufficient to decide the stable properties, and it surely loses some useful information at the horizon, which will be our further study; (3) if the Schwarzschild black hole exists stably, the other region not to communicate with it must exist, that is, the Kruskal space time is a real physical entity.

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